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Free vibration analysis of a uniform multi-span beam carrying multiple spring–mass systems

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Abstract

The literature regarding the free vibration analysis of single-span beams carrying a number of spring–mass systems is plenty, but that of multi-span beams carrying multiple spring–mass systems is fewer. Thus, this paper aims at determining the "exact" solutions for the natural frequencies and mode shapes of a uniform multi-span beam carrying multiple spring–mass systems. Firstly, the coefficient matrices for an intermediate pinned support, an intermediate spring–mass system, left-end support and right-end support of a uniform beam are derived. Next, the numerical assembly technique for the conventional finite element method is used to establish the overall coefficient matrix for the whole vibrating system. Finally, equating the last overall coefficient matrix to zero one determines the natural frequencies of the vibrating system and substituting the corresponding values of integration constants into the related eigenfunctions one determines the associated mode shapes. In this paper, the natural frequencies and associated mode shapes of the vibrating system are obtained directly from the differential equation of motion of the continuous beam and no other assumptions are made, thus, the last solutions are the exact ones. The effects of attached spring–mass systems on the free vibration characteristics of the 1–4-span beams are studied.

1. Introduction

The free vibration characteristics of a uniform beam carrying various concentrated elements (such as point masses, rotary inertias, linear springs, rotational springs, spring–mass systems, etc.) is an important problem in engineering, thus, a lot of reports have been published in this area. In Refs. [1–3], various techniques were presented to perform the free vibration analysis of beams carrying one or two concentrated elements. Wu et al. have found natural frequencies and mode shapes of a uniform beam carrying any number of rigidly attached point masses [4] and elastically attached point masses [5] by means of the analytical-and-numerical-combined method. Cha [6] solved the natural frequencies of a linear structure carrying any number of spring–mass systems using the assumed-modes method. Naguleswaran [7] found the natural frequencies of

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a Euler–Bernoulli beam with up to five elastic supports (including ends) using a fourth-order determinant equated to zero. By means of the Lagrange multipliers method, Gurgoze et al. [8–10] solved the eigenfrequencies of a cantilever beam with attached tip mass and a spring–mass system and studied the effect of an attached spring–mass system on the frequency spectrum of a cantilever beam. Furthermore, using the assumed modes method, they presented two alternative formulations of the frequency equation of a Bernoulli–Euler beam to which several spring–mass systems being attached in-span and then solved for the eigenfrequencies. Wu and Chou [11] obtained the exact solution of a single-span uniform beam carrying any number of spring–mass systems using the numerical assembly method (NAM).

From the above literature review one sees that the exact solutions for the natural frequencies and mode shapes of the ‘‘single-span’’ beams carrying either single or multiple spring–mass systems have been obtained [5,10,11]. However, the literature regarding the exact solutions for the natural frequencies and mode shapes of the ‘‘multi-span’’ beams carrying either single or multiple spring–mass systems have not yet been found. In Refs. [12,13], Gurgoze and Erol studied the forced vibration responses of a cantilever beam with single intermediate support, but they did not study the free vibration characteristic of the beam. In Refs. [11], Wu and Chou determined the exact natural frequencies and mode shapes of a ‘‘single-span’’ uniform beam carrying any number of spring–mass systems using the NAM. The present paper adopts the same method to investigate the free vibration characteristics of the ‘‘multi-span’’ uniform beam carrying multiple spring–mass systems.

2. Equation of motion and displacement function

Fig. 1 shows the sketch of a uniform beam supported by T pins (including those at the two ends of beam) and carrying S spring–mass systems. If each of the points that the T pinned supports or the S spring–mass systems located is called a ‘‘station’’, then the total number of stations is $N = S + T$. For convenience, three

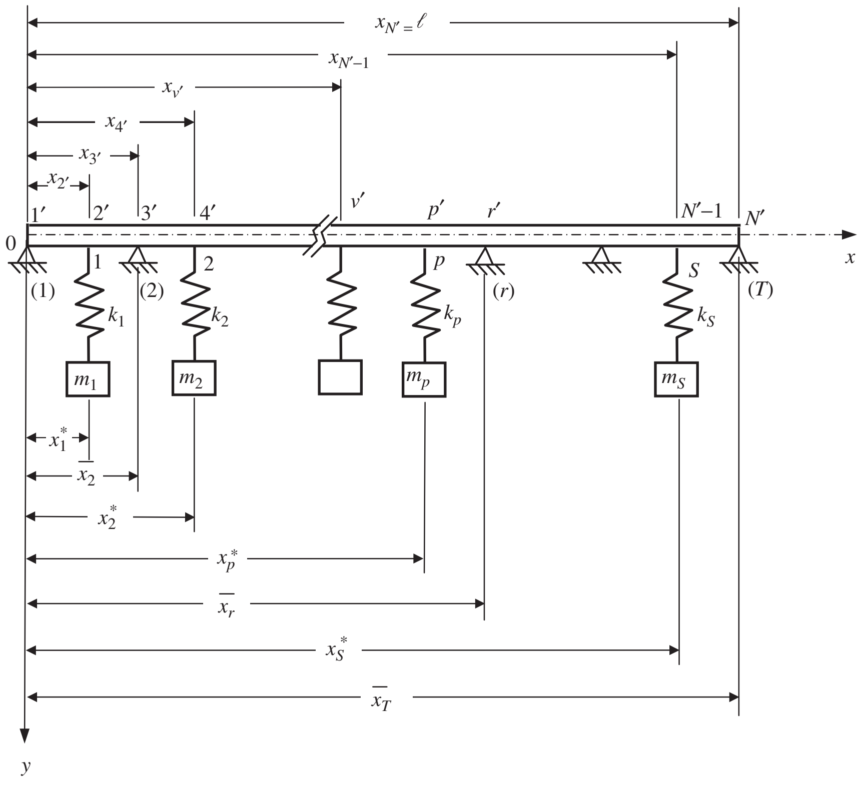

Fig. 1. Sketch for a uniform beam supported by T pins and carrying S spring–mass systems and the definitions for coordinates: $x_{v'}(v' = 1 \sim N)$ for stations, $x_p^*(p = 1 \sim S)$ for spring–mass systems and $\bar{x}_r(r = 1 \sim T)$ for pinned supports.

kinds of coordinates are used as one may see from Fig. 1. The positions for the stations are defined by $x_{v'}(v' = 1 \sim N)$, those for the spring–mass systems by $x_p^*(p = 1 \sim S)$ and those for the pinned supports by $\bar{x}_r(r = 1 \sim T)$. It is obvious that $x_{1'} = \bar{x}_1 = 0$ and $x_{N'} = \bar{x}_T = \ell$, because the first (left-end) support is at the origin of the coordinates and the final (right-end) support is at the other end of the beam with total length ℓ. For convenience, in Fig. 1, the numbers $1'$, $2'$, …, v', …, $N'-1$ and N' above the x-axis refer to the numbering of stations, while the symbols of 1, 2, …, p, …, and S, and those of (1), (2), …, (r), …, and (T), below the x-axis refer to the numbering of spring–mass systems and pinned supports, respectively. It is noted that the support numbers are enclosed in parentheses ( ), while the spring–mass system numbers are not.

For a uniform Euler–Bernoulli beam, its equation of motion is given by

$$EI\,\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\,\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \tag{1}$$

where E is Young's modulus, I is moment of inertia of the cross-sectional area, $\bar{m}$ is mass per unit length of the beam, $y(x,t)$ is transverse deflection of the beam at position x and time t.

For the pth spring–mass system, its equation of motion is given by

$$m_p \ddot{z}_p + k_p(z_p - y_p) = 0, \tag{2}$$

where m_p and k_p denote the point mass and spring constant of the pth spring–mass system, respectively, z_p and $\ddot{z}_p$ denote the displacement and acceleration of the pth spring mass (m_p) relative to its static equilibrium position, respectively, while y_p is the transverse displacement of the beam at the attaching point of the pth spring–mass system (cf. Fig. 1).

When the whole vibrating system shown in Fig. 1 performs harmonic free vibration, one has

$$y(x,t) = Y(x)\mathrm{e}^{\mathrm{j}\bar{\omega}t} \tag{3}$$

and

$$z_p(t) = Z_p \mathrm{e}^{\mathrm{j}\bar{\omega}t}, \tag{4}$$

where $Y(x)$ and Z_p are the amplitudes of $y(x, t)$ and $z_p(t)$, respectively, $\bar{\omega}$ is the natural frequency of the whole vibrating system and $\mathrm{j} = \sqrt{-1}$.

The substitution of Eq. (3) into Eq. (1) gives

$$Y'''' - \beta^4 Y = 0, \tag{5}$$

where

$$\beta^4 = \frac{\bar{\omega}^2 \bar{m}}{EI} \tag{6a}$$

or

$$\bar{\omega} = (\beta\ell)^2 \left(\frac{EI}{\bar{m}\ell^4}\right)^{1/2}. \tag{6b}$$

The solution of Eq. (5) takes the form

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x. \tag{7}$$

This equation is the displacement function for each beam segment between any two adjacent stations of the multi-span beam shown in Fig. 1.

3. Determination of natural frequencies and mode shapes

For an arbitrary point located at $x_{v'}$ (cf. Fig. 1), from Eq. (7) one obtains

$$Y_{v'}(\xi_{v'}) = C_{v',1} \sin \Omega\xi_{v'} + C_{v',2} \cos \Omega\xi_{v'} + C_{v',3} \sinh \Omega\xi_{v'} + C_{v',4} \cosh \Omega\xi_{v'}, \tag{8}$$

$$Y'_{v'}(\xi_{v'}) = \Omega C_{v',1} \cos \Omega\xi_{v'} - \Omega C_{v',2} \sin \Omega\xi_{v} + \Omega C_{v',3} \cosh \Omega\xi_{v'} + \Omega C_{v',4} \sinh \Omega\xi_{v'}, \tag{9}$$

$$Y''_{v'}(\xi_{v'}) = -\Omega^2 C_{v',1} \sin \Omega\xi_{v'} - \Omega^2 C_{v',2} \cos \Omega\xi_{v'} + \Omega^2 C_{v',3} \sinh \Omega\xi_{v'} + \Omega^2 C_{v',4} \cosh \Omega\xi_{v'}, \tag{10}$$

$$Y'''_{v'}(\xi_{v'}) = -\Omega^3 C_{v',1} \cos \Omega\xi_{v'} + \Omega^3 C_{v',2} \sin \Omega\xi_{v'} + \Omega^3 C_{v',3} \cosh \Omega\xi_{v'} + \Omega^3 C_{v',4} \sinh \Omega\xi_{v'} \tag{11}$$

with

$$\xi_{v'} = \frac{x_{v'}}{\ell}, \tag{12}$$

$$\Omega = \beta\ell. \tag{13}$$

If the left-end support (i.e., station 1′) of the beam is pinned as shown Fig. 1, then the boundary conditions are

$$Y_{1'}(0) = Y''_{1'}(0) = 0, \tag{14a,b}$$

where the primes refer to differentiations with respect to the coordinate x.

From Eqs. (8), (10) and (14), one obtains

$$C_{1',2} + C_{1',4} = 0, \tag{15a}$$

$$-C_{1',2} + C_{1',4} = 0 \tag{15b}$$

or in matrix form

$$[B_{1'}]\{C_{1'}\} = 0, \tag{16}$$

where

$$\begin{matrix} & 1 & 2 & 3 & 4 & \\ [B_{1'}] = & \Bigg[\begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 1 \end{matrix} \Bigg] & \begin{matrix} 1 \\ 2 \end{matrix}, \end{matrix} \tag{17}$$

$$\{C_{1'}\} = \{C_{1',1}\ \ C_{1',2}\ \ C_{1',3}\ \ C_{1',4}\}. \tag{18}$$

If the station numbering corresponding to the pth intermediate spring–mass system is represented by p', then the continuity of deformations and equilibrium of moments and forces require that

$$Y^L_{p'}(\xi_{p'}) = Y^R_{p'}(\xi_{p'}), \tag{19a}$$

$$Y'^L_{p'}(\xi_{p'}) = Y'^R_{p'}(\xi_{p'}), \tag{19b}$$

$$Y''^L_{p'}(\xi_{p'}) = Y''^R_{p'}(\xi_{p'}), \tag{19c}$$

$$Y'''^L_{p'}(\xi_{p'}) + m_p/EI\bar{\omega}^2 Z_p - Y'''^R_{p'}(\xi_{p'}) = 0 \tag{19d}$$

with

$$\xi_{p'} = \frac{x_{p'}}{\ell}, \quad \hat{m}_p = \frac{m_p}{\bar{m}\ell}. \tag{19e,f}$$

In Eqs. (19a)–(19d), the superscripts "L" and "R" refer to the left side and right side of station p', respectively.

The substitution of Eqs. (3) and (4) into Eq. (2) gives

$$k_p Y_{p'} - (k_p - m_p\bar{\omega}^2)Z_p = 0 \tag{20a}$$

or

$$Y_{p'} + (\lambda_p^2 - 1)Z_p = 0, \tag{20b}$$

where

$$\lambda_p = \bar{\omega}/\omega_p \tag{21}$$

with

$$\omega_p = \sqrt{k_p/m_p}. \tag{22}$$

In the last expressions, $x_{p'}$ is the coordinate of station p' and ω_p defined by Eq. (22) denotes the natural frequency of the pth spring–mass system with respect to the static beam.

Substituting Eqs. (8)–(11) into Eqs. (19a)–(19d), one obtains

$$\begin{aligned} &C_{p'-1,1} \sin \Omega\xi_{p'} + C_{p'-1,2} \cos \Omega\xi_{p'} + C_{p'-1,3} \sinh \Omega\xi_{p'} + C_{p'-1,4} \cosh \Omega\xi_{p'} \\ &\quad - C_{p',1} \sin \Omega\xi_{p'} - C_{p',2} \cos \Omega\xi_{p'} - C_{p',3} \sinh \Omega\xi_{p'} - C_{p',4} \cosh \Omega\xi_{p'} = 0 \end{aligned} \tag{23a}$$

$$\begin{aligned} &C_{p'-1,1} \cos \Omega\xi_{p'} - C_{p'-1,2} \sin \Omega\xi_{p'} + C_{p'-1,3} \cosh \Omega\xi_{p'} + C_{p'-1,4} \sinh \Omega\xi_{p'} \\ &\quad - C_{p',1} \cos \Omega\xi_{p'} + C_{p',2} \sin \Omega\xi_{p'} - C_{p',3} \cosh \Omega\xi_{p'} - C_{p',4} \sinh \Omega\xi_{p'} = 0, \end{aligned} \tag{23b}$$

$$\begin{aligned} &- C_{p'-1,1} \sin \Omega\xi_{p'} - C_{p'-1,2} \cos \Omega\xi_{p'} + C_{p'-1,3} \sinh \Omega\xi_{p'} + C_{p'-1,4} \cosh \Omega\xi_{p'} \\ &\quad + C_{p',1} \sin \Omega\xi_{p'} + C_{p',2} \cos \Omega\xi_{p'} - C_{p',3} \sinh \Omega\xi_{p'} - C_{p',4} \cosh \Omega\xi_{p'} = 0 \end{aligned} \tag{23c}$$

$$\begin{aligned} &- C_{p'-1,1} \cos \Omega\xi_{p'} + C_{p'-1,1} \sin \Omega\xi_{p'} + C_{p'-1,1} \cosh \Omega\xi_{p'} + C_{p'-1,1} \sinh \Omega\xi_{p'} + \hat{m}_p\Omega Z_p \\ &\quad + C_{p',1} \cos \Omega\xi_{p'} - C_{p',2} \sin \Omega\xi_{p'} - C_{p',3} \cosh \Omega\xi_{p'} - C_{p',4} \sinh \Omega\xi_{p'} = 0. \end{aligned} \tag{23d}$$

The substitution of Eq. (8) into Eq. (20b) leads to

$$C_{p',1} \sin \Omega\xi_{p'} + C_{p',2} \cos \Omega\xi_{p'} + C_{p',3} \sinh \Omega\xi_{p'} + C_{p',4} \cosh \Omega\xi_{p'} + (\lambda_p^2 - 1)Z_p = 0. \tag{23e}$$

Writing Eqs. (23a)–(23e) in matrix form, one obtains

$$[B_{p'}]\{C_{p'}\} = 0, \tag{24}$$

where

$$[B_{p'}] = \begin{matrix} & \begin{matrix} 4p'-3 & 4p'-2 & 4p'-1 & 4p' & 4p'+1 & 4p'+2 & 4p'+3 & 4p'+4 & 4p'+5 \end{matrix} & \\ & \begin{bmatrix} s\theta_{p'} & c\theta_{p'} & sh\theta_{p'} & ch\theta_{p'} & -s\theta_{p'} & -c\theta_{p'} & -sh\theta_{p'} & -ch\theta_{p'} & 0 \\ c\theta_{p'} & -s\theta_{p'} & ch\theta_{p'} & sh\theta_{p'} & -c\theta_{p'} & s\theta_{p'} & -ch\theta_{p'} & -sh\theta_{p'} & 0 \\ -s\theta_{p'} & -c\theta_{p'} & sh\theta_{p'} & ch\theta_{p'} & s\theta_{p'} & c\theta_{p'} & -sh\theta_{p'} & -ch\theta_{p'} & 0 \\ -c\theta_{p'} & s\theta_{p'} & ch\theta_{p'} & sh\theta_{p'} & c\theta_{p'} & -s\theta_{p'} & -ch\theta_{p'} & -sh\theta_{p'} & \hat{m}_p\Omega \\ s\theta_{p'} & c\theta_{p'} & sh\theta_{p'} & ch\theta_{p'} & 0 & 0 & 0 & 0 & \lambda_p^2 - 1 \end{bmatrix} & \begin{matrix} 4p'-1 \\ 4p' \\ 4p'+1, \\ 4p'+2 \\ 4p'+3 \end{matrix} \end{matrix} \tag{25}$$

$$\{C_{p'}\} = \{\, C_{p'-1,1} \quad C_{p'-1,2} \quad C_{p'-1,3} \quad C_{p'-1,4} \quad C_{p',1} \quad C_{p',2} \quad C_{p',3} \quad C_{p',4} \quad Z_p \,\}. \tag{26}$$

The symbols appearing in Eq. (25) are defined by

$$\theta_{p'} = \Omega\xi_{p'},\ \ s\theta_{p'} = \sin \Omega\xi_{p'},\ \ c\theta_{p'} = \cos \Omega\xi_{p'},\ \ sh\theta_{p'} = \sinh \Omega\xi_{p'},\ \ ch\theta_{p'} = \cosh \Omega\xi_{p'}. \tag{27}$$

Similarly, if the station numbering corresponding to the (r)th intermediate support is represented by r', then the continuity of deformations and equilibrium of moments and forces require that

$$Y_{r'}^{L}(\xi_{r'}) = Y_{r'}^{R}(\xi_{r'}) = 0, \tag{28a,b}$$

$$Y'^{L}_{r'}(\xi_{r'}) = Y'^{R}_{r'}(\xi_{r'}), \tag{28c}$$

$$Y''^{L}_{r'}(\xi_{r'}) = Y''^{R}_{r'}(\xi_{r'}) \tag{28d}$$

with

$$\xi_{r'} = \frac{x_{r'}}{\ell}, \tag{29}$$

where $x_{r'}$ is the coordinate of station r' at which the (r)th intermediate support is located.

Introducing Eqs. (8)–(11) into Eqs. (28), one obtains

$$C_{r'-1,1} \sin \Omega\xi_{r'} + C_{r'-1,2} \cos \Omega\xi_{r'} + C_{r'-1,3} \sinh \Omega\xi_{r'} + C_{r'-1,4} \cosh \Omega\xi_{r'} = 0, \tag{30a}$$

$$C_{r',1} \sin \Omega\xi_{r'} + C_{r',2} \cos \Omega\xi_{r'} + C_{r',3} \sinh \Omega\xi_{r'} + C_{r',4} \cosh \Omega\xi_{r'} = 0, \tag{30b}$$

$$\begin{aligned} &C_{r'-1,1} \cos \Omega\xi_{r'} - C_{r'-1,2} \sin \Omega\xi_{r'} + C_{r'-1,3} \cosh \Omega\xi_{r'} + C_{r'-1,4} \sinh \Omega\xi_{r'} \\ &- C_{r',1} \cos \Omega\xi_{r'} + C_{r',2} \sin \Omega\xi_{r'} - C_{r',3} \cosh \Omega\xi_{r'} - C_{r',4} \sinh \Omega\xi_{r'} = 0, \end{aligned} \tag{30c}$$

$$\begin{aligned} &- C_{r'-1,1} \sin \Omega\xi_{r'} - C_{r'-1,2} \cos \Omega\xi_{r'} + C_{r'-1,3} \sinh \Omega\xi_{r'} + C_{r'-1,4} \cosh \Omega\xi_{r'} \\ &+ C_{r',1} \sin \Omega\xi_{r'} + C_{r',2} \cos \Omega\xi_{r'} - C_{r',3} \sinh \Omega\xi_{r'} - C_{r',4} \cosh \Omega\xi_{r'} = 0 \end{aligned} \tag{30d}$$

or

$$[B_{r'}]\{C_{r'}\} = 0, \tag{31}$$

where

$$[B_{r'}] = \begin{array}{c} \begin{matrix} 4r'-3 & 4r'-2 & 4r'-1 & 4r' & 4r'+1 & 4r'+2 & 4r'+3 & 4r'+4 \end{matrix} \\ \left[\begin{matrix} s\theta_{r'} & c\theta_{r'} & sh\theta_{r'} & ch\theta_{r'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\theta_{r'} & c\theta_{r'} & sh\theta_{r'} & ch\theta_{r'} \\ c\theta_{r'} & -s\theta_{r'} & ch\theta_{r'} & sh\theta_{r'} & -c\theta_{r'} & s\theta_{r'} & -ch\theta_{r'} & -sh\theta_{r'} \\ -s\theta_{r'} & -c\theta_{r'} & sh\theta_{r'} & ch\theta_{r'} & s\theta_{r'} & c\theta_{r'} & -sh\theta_{r'} & -ch\theta_{r'} \end{matrix}\right] \end{array} \begin{matrix} 4r'-1 \\ 4r' \\ 4r'+1 \\ 4r'+2 \end{matrix}, \tag{32}$$

$$\{C_{r'}\} = \{ C_{r'-1,1} \quad C_{r'-1,2} \quad C_{r'-1,3} \quad C_{r'-1,4} \quad C_{r',1} \quad C_{r',2} \quad C_{r',3} \quad C_{r',4} \}, \tag{33}$$

where

$$\theta_{r'} = \Omega\xi_{r'}, \;\; s\theta_{r'} = \sin \Omega\xi_{r'}, \;\; c\theta_{r'} = \cos \Omega\xi_{r'}, \;\; sh\theta_{r'} = \sinh \Omega\xi_{r'}, \;\; ch\theta_{r'} = \cosh \Omega\xi_{r'}. \tag{34}$$

From Fig. 1, one sees that the right-end support (i.e., station N') of the beam is pinned, thus the boundary conditions are

$$Y_{N'}(\ell) = Y''_{N'}(\ell) = 0. \tag{35a,b}$$

Substituting Eqs. (8) and (10) into Eqs. (35) gives

$$C_{N',1} \sin \Omega + C_{N',2} \cos \Omega + C_{N',3} \sinh \Omega + C_{N',4} \cosh \Omega = 0, \tag{36a}$$

$$-C_{N',1} \sin \Omega - C_{N',2} \cos \Omega + C_{N',3} \sinh \Omega + C_{N',4} \cosh \Omega = 0 \tag{36b}$$

or

$$[B_{N'}]\{C_{N'}\} = 0, \tag{37}$$

where

$$[B_{N'}] = \begin{array}{c} \begin{matrix} 4N'_i+1 & 4N'_i+2 & 4N'_i+3 & 4N'_i+4 \end{matrix} \\ \left[\begin{matrix} \sin \Omega & \cos \Omega & \sinh \Omega & \cosh \Omega \\ -\sin \Omega & -\cos \Omega & \sinh \Omega & \cosh \Omega \end{matrix}\right] \end{array} \begin{matrix} q-1 \\ q \end{matrix}, \tag{38}$$

$$\{C_{N'}\} = \{ C_{N',1} \quad C_{N',2} \quad C_{N',3} \quad C_{N',4} \}. \tag{39}$$

In Eq. (38), N'_i denotes the total number of intermediate stations given by (cf. Fig. 1)

$$N'_i = N' - 2 \tag{40a}$$

with

$$N' = S + T \tag{40b}$$

and q denotes the total number of equations for the integration constants given by

$$q = 4(T-2) + 5S + 4. \tag{41}$$

From the above derivations one sees that one may obtain four equations from each intermediate station at which a pinned support is located, five equations from each intermediate station at which a spring–mass system is located and two equations from either the left-end station or the right-end station of the beam. Therefore, the total number of equations for the integration constants is $q = 4(T-2) + 5S + 4$. Note that, in Eq. (40), N' is the total number of stations as one may see from Fig. 1.

The integration constants relating to the left-end support (i.e., station $1'$) and those relating to the right-end support (i.e., station N') of the beam are determined by Eqs. (16) and (37), respectively, while those relating to the intermediate stations (i.e., stations 2 to $N'-1$) are determined by Eqs. (24) or (31) depending upon the spring–mass system or pinned support being located there. The associated coefficient matrices are given by $[B_{1'}], [B_{p'}], [B_{r'}]$ and $[B_{N'}]$ as one may see from Eqs. (17), (25), (32) and (38), respectively. From the last four equations one may see that the identification number for each element of the last four coefficient matrices is shown on the top side and right side of each matrix. Therefore, using the numerical assembly technique as done by the conventional finite element method (FEM) one may obtain a matrix equation for all the integration constants of the entire beam

$$[\bar{B}]\{\bar{C}\} = 0. \tag{42}$$

Non-trivial solution of Eq. (42) requires that

$$|\bar{B}| = 0, \tag{43}$$

which is the frequency equation for the present problem.

In this paper, the half-interval method [14–16] is used to find the natural frequencies of the multi-span beam carrying multiple spring–mass systems, $\bar{\omega}_i (i = 1, 2, \ldots)$. For each natural frequency $\bar{\omega}_i$, one may obtain the corresponding integration constants from Eq. (42). The substitution of the last integration constants into the displacement functions of the associated beam segments will determine the corresponding mode shape of the beam, $Y^{(i)}(\xi)$. For reference, the overall coefficient matrix $[\bar{B}]$ for a two-span uniform beam with one intermediate spring–mass system (or a pinned–pinned uniform beam with one intermediate pinned support and one intermediate spring–mass system) is shown in the appendix.

The lowest several natural frequencies $\bar{\omega}_i (i = 1, 2, \ldots)$ are determined as follows: For the first step, a trial dimensionless frequency coefficient value of $\Omega = \Omega_r$ is assumed and the numerical values of all coefficients $\bar{B}_{k\ell}(k, \ell = q)$ of the determinant $|\bar{B}|_{q\times q}$ corresponding to $\Omega_r (r = 1, 2, 3, \ldots)$ are calculated. Once the values of $\bar{B}_{k\ell}(k, \ell = q)$ are obtained, one may determine the value of $|\bar{B}|_{q\times q}$ using the computer *subroutine* for determinant and let $D_r = |\bar{B}(\Omega_r)|_{q\times q}$. For the next step, a new trial value, $\Omega = \Omega_{r+1} = \Omega_r + \Delta\Omega$ with $\Delta\Omega$ representing the increment of Ω, is assumed, and the same calculations are repeated to determine the new value of determinant corresponding to Ω_{r+1}, i.e., $D_{r+1} = |\bar{B}(\Omega_{r+1})|_{q\times q}$. If D_r and D_{r+1} have opposite signs, then there is at least one dimensionless frequency coefficient in the interval (Ω_r, Ω_{r+1}). In order to avoid missing some dimensionless frequency coefficient, the increment $\Delta\Omega$ should be small enough. If the increment $\Delta\Omega$ is small enough (e.g., $\Delta\Omega = 0.01$), it is likely to contain only one dimensionless frequency coefficient $\bar{\Omega}$. Because one can obtain only one value of dimensionless frequency coefficient each time D_r and D_{r+1} have opposite signs, one needs to continue the above-mentioned procedures until the entire lowest several intervals are obtained. After finding all intervals, accurate values of $\bar{\Omega}_i$ are calculated respectively using the well-known half-interval method [14–16]. Besides, checking with the results of FEM is also used to ensure no natural frequencies $\bar{\omega}_i$ being missed in this paper. It is noted that the relationship between $\Omega_i (= \bar{\Omega}_i)$ and $\bar{\omega}_i$ is given by Eqs. (13) and (6b).

4. Numerical results and discussions

Before the free vibration analysis of a multi-span uniform beam carrying multiple spring–mass systems is performed, the reliability of the theory and the computer program developed for this paper are confirmed by comparing the present results with those obtained from the existing literature or the conventional FEM. Unless particularly mentioned, all the numerical results of this paper are obtained based on a uniform Euler–Bernoulli beam with the following given data: Young's modulus $E = 2.069 \times 10^{11}\,\mathrm{N/m^2}$, diameter $d = 0.05\,\mathrm{m}$, moment of inertia of the cross-sectional area $I = 3.06796 \times 10^{-7}\,\mathrm{m^4}$, mass per unit length $\bar{m} = 15.3875\,\mathrm{kg/m}$ and total length $\ell = 1\,\mathrm{m}$, total mass $m_b = \bar{m}\ell = 15.3875\,\mathrm{kg}$, reference spring constant $k_b = EI/\ell^3 = 6.34761 \times 10^4\,\mathrm{N/m}$. For convenience, two non-dimensional parameters, $\hat{m}_p$ and $\hat{k}_p$, are introduced, they are defined by $\hat{m}_p = m_p/m_b$ and $\hat{k}_p = k_p/k_b$. Besides, in FEM, the two-node beam elements are used and the entire continuous beam is subdivided into 40 beam elements. Since each node has two degrees of freedom (dofs), the total dof for the entire unconstrained beam is $2(40+1) = 82$.

4.1. Reliability of the developed computer program

For convenience, in this paper, the pinned–pinned beam is called a "bare" beam if it does not carry any spring–mass systems and is called a "loaded" beam if it carries any number of spring–mass systems. The first example (see Fig. 2) studied is the uniform single-span pinned–pinned beam carrying one to five intermediate spring–mass systems. Table 1 shows the natural frequencies of the five spring–mass systems with respect to the static beam defined by Eq. (22), ω_p (rad/s), with the subscript p denoting the numbering of five spring–mass systems, and $\hat{m}_p = m_p/m_b$ and $\hat{k}_p = k_p/k_b$ ($p = 1$ to 5) denoting the associated non-dimensional lumped masses and spring constants, respectively. Table 2 shows the lowest five natural frequencies of the "loaded" beam for three cases: (1) The 1st spring–mass system (i.e., $p = 1$) of Table 1 is attached on the beam at $\xi_p^* = x_p^*/\ell = 0.75$; (2) The 1st, 3rd and 5th spring–mass systems (i.e., $p = 1, 3, 5$) of Table 1 are attached at $\xi_p^* = 0.1$, 0.4 and 0.8, respectively; (3) The 1st, 2nd, 3rd, 4th and 5th spring–mass systems (i.e., $p = 1$ to 5) are attached at $\xi_p^* = 0.1$, 0.2, 0.4, 0.6 and 0.8, respectively. The lowest five natural frequencies of the loaded beam for the last three cases are listed in the 1st, 3rd and 5th rows of results of Table 2, respectively, and the corresponding ones obtained from Ref. [11] are listed in the 2nd, 4th and 6th rows of results of Table 2. For comparison, the lowest five "exact" natural frequencies of the single-span "bare" pinned–pinned beam, ω_{bi} ($i = 1$ to 5), are also presented next Table 2: 633.9001, 2535.6003, 5705.1007, 10142.4012, 15847.5019 rad/s. From Table 2 one sees that the results of this paper are in good agreement with those of Ref. [11]. Furthermore, from Tables 1 and 2 one sees that $\bar{\omega}_1 = 243.8580 \approx \omega_1 = 248.7521$ rad/s for case (1),

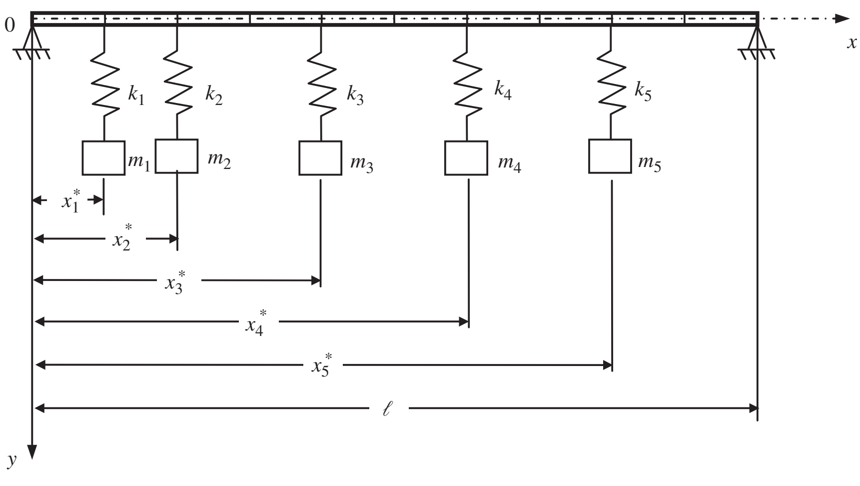

Fig. 2. A single-span pinned–pinned beam carrying five intermediate spring–mass systems with $\hat{m}_p = m_p/m_b$ and $\hat{k}_p = k_p/k_b$ located at x_p^*, $p = 1$ to 5.

$\bar{\omega}_1 = 152.7341 \approx \omega_5 = 157.3246\,\mathrm{rad/s}$, $\bar{\omega}_2 = 185.0951 \approx \omega_3 = 192.6825\,\mathrm{rad/s}$ and $\bar{\omega}_3 = 247.8314 \approx \omega_1 = 248.7521\,\mathrm{rad/s}$ for case (2), and $\bar{\omega}_i \approx \omega_{6-i}$ ($i = 1$ to 5) for case (3). In other words, the lowest p natural frequencies of the loaded beam, $\bar{\omega}_p (p = 1, 2, 3, \ldots)$, are very close to the natural frequencies of the attached p spring–mass systems with respect to the static beam, $\omega_p (p = 1, 2, 3, \ldots)$. It is also seen that $\bar{\omega}_i \approx \omega_{b,i-1}$ with $i = 2$ to 5 for case (1) and $\bar{\omega}_i \approx \omega_{b,i-3}$ with $i = 4,5$ for case (2), where ω_{bi} denotes the ith natural frequency of the "bare" beam listed next Table 2.

Table 1
Natural frequencies of the five spring–mass systems with respect to the static beam, ω_p (rad/s), respectively

Numbering, p	1	2	3	4	5
$\hat{m}_p = m_p/m_b$	0.2	0.3	0.5	0.65	1.0
$\hat{k}_p = k_p/k_b$	3.0	3.5	4.5	5.0	6.0
$\omega_p = \sqrt{k_p/m_p}$	248.7521	219.3787	192.6825	178.1351	157.3246

Table 2
Influence of the spring–mass systems shown in Table 1 on the lowest five natural frequencies of the single-span pinned–pinned beam shown in Fig. 2

Cases	Locations $\xi_P^* = x_p^*/\ell$	Methods	Natural frequencies, $\bar{\omega}_i$ (rad/s)				
			$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
1	0.75 ($p = 1$)	Present	243.8577	645.2028	2540.5298	5706.1880	10142.4002
		Ref. [11]	243.8579	645.2030	2540.5306	5706.1886	10142.4012
2	0.1, 0.4, 0.8 ($p = 1$, 3, 5)	Present	152.7339	185.0949	247.8313	677.5959	2548.6572
		Ref. [11]	152.7341	185.0950	247.8314	677.5961	2548.6577
3	0.1, 0.2, 0.4, 0.6, 0.8 ($p = 1$ to 5)	Present	150.9571	169.4728	187.9146	217.1278	247.9867
		Ref. [11]	150.9571	169.4729	187.9147	217.1279	247.9868

Note: the lowest five "exact" natural frequencies of the single-span "bare" pinned–pinned beam, ω_{bi} ($i = 1$ to 5), are 633.9001, 2535.6003, 5705.1007, 10142.4012, 15847.5019 rad/s.

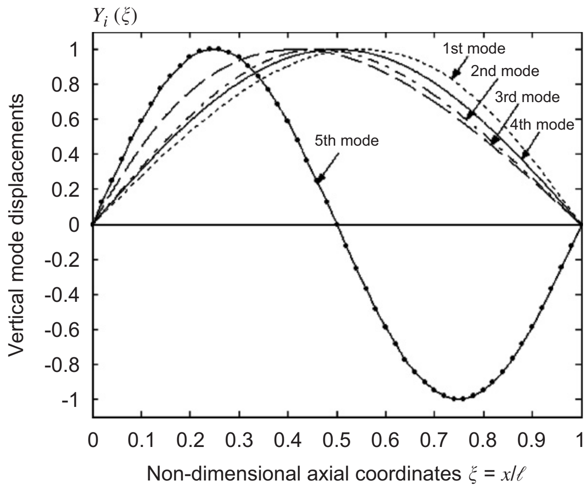

Fig. 3. The lowest five mode shapes of a uniform single-span pinned–pinned beam carrying three intermediate spring–mass systems with ($\hat{m}_1 = 0.2$, $\hat{k}_1 = 3.0$), ($\hat{m}_3 = 0.5$, $\hat{k}_3 = 4.5$) and ($\hat{m}_5 = 1.0$, $\hat{k}_5 = 6.0$) located at $\xi_1^* = x_1^*/\ell = 0.1$, $\xi_3^* = 0.4$ and $\xi_5^* = 0.8$, respectively.

The lowest five mode shapes of the loaded beam for case (2) are shown in Fig. 3. In which, the 1st, 2nd, 3rd, 4th and 5th mode shapes are represented by the curves. ----, ———, ———, ———, and —·—·, respectively. It is seen that the lowest three mode shapes take the same form and are similar to the 4th one, this is because the 4th natural frequency of the loaded beam is very close to the 1st one of the "bare" beam (i.e., $\bar{\omega}_4 = 677.5963 \approx \omega_{b1} = 633.9030\,\text{rad/s}$) and so is the corresponding mode shapes, in addition, the 1st, 2nd and 3rd mode shapes of the loaded beam are related to the free vibrations of the three spring–mass systems with respect to the static beam, respectively. It is evident that the 5th mode shape of the loaded beam as shown in Fig. 3 will be very close to the 2nd mode shape of the bare beam, because $\bar{\omega}_5 = 2548.6581 \approx \omega_{b2} = 2535.6119\,\text{rad/s}$.

4.2. Free vibration analysis of a two-span beam carrying one spring–mass system

In the last subsection, the pinned–pinned beam is single-span and is attached by one to five intermediate spring–mass systems. However, the beam studied in this subsection is two-span (pinned at its two ends and at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$) and carries one intermediate spring–mass system at $\xi_1^* = 0.75$ as shown in Fig. 4. The non-dimensional magnitude of the lumped mass is $\hat{m}_1 = m_1/m_b = 0.2$ and that of the spring constant is $\hat{k}_1 = k_1/k_b = 3.0$. For reference, the explicit form of the overall coefficient matrix $[B]$ for the present example is given in the appendix at the end of this paper. Although it is only a 13×13 square matrix, it is too complicated to be listed. The lowest five natural frequencies of the "loaded" beam are shown in the 1st row of results of Table 3 and the associated mode shapes are plotted in Fig. 5. For comparison, the lowest five natural frequencies of the "bare" beam (i.e., $p = 0$) are also listed in the 3rd row of results of Table 3. In Fig. 5, the 1st, 2nd, 3rd, 4th and 5th mode shapes are represented by the curves, ..., ———, ———, ———, and –·–·–·, respectively. It is seen that the 1st mode shape and the 2nd one take the same form. It is similar to the single-span beam studied in the last subsection that $\bar{\omega}_1 = 248.1408 \approx \omega_1 = 248.7521\,\text{rad/s}$ and $\bar{\omega}_i \approx \omega_{b,i-1}$ for $i = 2$ to 5, where $\bar{\omega}_i$ denotes the ith natural frequency of "loaded"

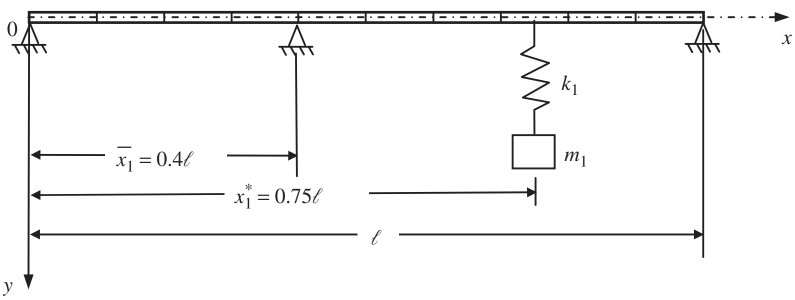

Fig. 4. A uniform two-span pinned–pinned beam (with one intermediate support at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4 = 0.4$) carrying one intermediate spring–mass system with $\hat{m}_1 = 0.2$ and $\hat{k}_1 = 3.0$ located at $\xi_1^* = x_1^*/\ell = 0.75$.

Table 3
Influence of the intermediate spring–mass system on the lowest five natural frequencies of the two-span pinned–pinned beam shown in Fig. 4

No. of spring–mass system, p	Methods	Natural frequencies (rad/s)				
		$\bar{\omega}_1$ (or ω_{b1})	$\bar{\omega}_2$ (or ω_{b2})	$\bar{\omega}_3$ (or ω_{b3})	$\bar{\omega}_4$ (or ω_{b4})	$\bar{\omega}_5$ (or ω_{b5})
1	Present	247.6358	2156.8778	4938.1451	8220.1624	15847.8928
	FEM	247.6360	2156.8780	4938.1520	8220.1982	15848.1531
0	Present	2147.6735	4937.5110	8219.9705	15847.5054	19288.5388
	FEM	2147.6740	4937.5168	8220.0047	15847.7625	19289.0078

Note: $\bar{\omega}_i$ denotes the ith natural frequency of "loaded" beam and ω_{bi} denotes that of the "bare" beam.

beam and ω_{bi} denotes that of "bare" beam. In addition to the results of the present NAM, those of the FEM are also given in the 2nd and 4th rows of results of Table 3. It is seen that good agreement between the corresponding results is achieved.

4.3. Free vibration analysis of a three-span beam carrying three spring–mass systems

In this subsection, a *three-span* pinned–pinned beam carrying three intermediate spring–mass systems as shown in Fig. 6 is studied. The locations of the *two* intermediate pinned supports are at $\bar{x}_1 = 0.3\ell$ and $\bar{x}_2 = 0.7\ell$, while those of the three spring–mass systems are at $x_1^* = 0.1\ell$, $x_2^* = 0.4\ell$ and $x_3^* = 0.8\ell$. The non-dimensional magnitudes of the three lumped masses are $\hat{m}_1 = 0.2$, $\hat{m}_2 = 0.3$ and $\hat{m}_3 = 0.5$, while those of the three spring constants are $\hat{k}_1 = 3.0$, $\hat{k}_2 = 3.5$ and $\hat{k}_3 = 4.5$, respectively. The lowest five natural frequencies of the loaded beam are shown in the 1st and 2nd rows of Table 4 and the associated mode shapes are shown in Fig. 7. In the last figure, the 1st, 2nd, 3rd, 4th and 5th mode shapes are represented by the curves,

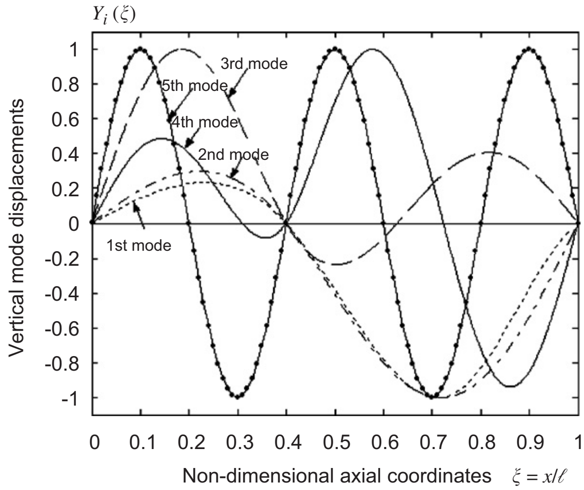

Fig. 5. The lowest five mode shapes of a two-span uniform pinned–pinned beam (with one intermediate support located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$) carrying one intermediate spring–mass system with $\hat{m}_1 = 0.2$ and $\hat{k}_1 = 3.0$ located at $\xi_1^* = x_1^*/\ell = 0.75$.

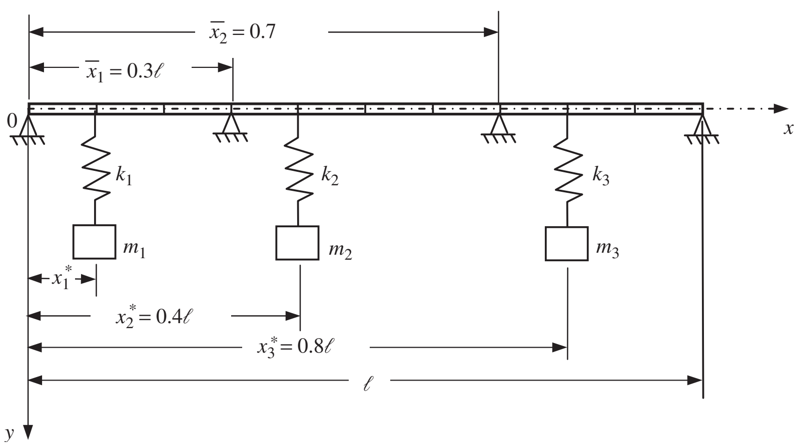

Fig. 6. A three-span pinned–pinned beam (with two intermediate supports at $\bar{x}_1 = 0.3\ell$ and $\bar{x}_2 = 0.7\ell$) carrying three intermediate spring–mass systems located at $x_1^* = 0.1\ell$, $x_2^* = 0.4\ell$ and $x_3^* = 0.8\ell$.

Table 4
Influence of the three intermediate spring–mass systems on the lowest five natural frequencies of the three-span pinned–pinned beam shown in Fig. 6

No. of spring–mass system, p	Methods	Natural frequencies (rad/s)				
		$\bar{\omega}_1$ (or ω_{b1})	$\bar{\omega}_2$ (or ω_{b2})	$\bar{\omega}_3$ (or ω_{b3})	$\bar{\omega}_4$ (or ω_{b4})	$\bar{\omega}_5$ (or ω_{b5})
3	Present	192.5522	219.2418	248.6207	5252.5167	8595.0801
	FEM	192.5523	219.2419	248.6207	5252.5272	8595.1228
0	Present	5248.1555	8590.6934	10305.8151	19750.4270	30163.2250
	FEM	5248.1650	8590.7350	10305.8870	19750.5396	30163.4182

Note: $\bar{\omega}_i$ denotes the ith natural frequency of "loaded" beam and ω_{bi} denotes that of "bare" beam.

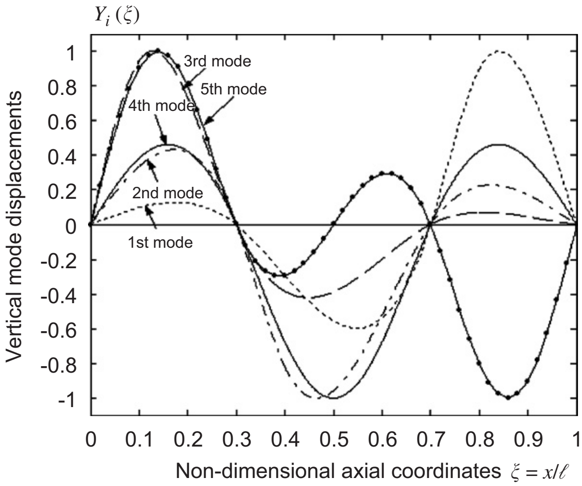

Fig. 7. The lowest five mode shapes of a three-span pinned–pinned beam (with two intermediate supports at $\bar{x}_1 = 0.3\ell$ and $\bar{x}_2 = 0.7\ell$) carrying three intermediate spring–mass systems located at $x_1^* = 0.1\ell$, $x_2^* = 0.4\ell$ and $x_3^* = 0.8\ell$. The non-dimensional magnitudes of the three spring–mass systems are $\hat{m}_1 = 0.2$, $\hat{m}_2 = 0.3$, $\hat{m}_3 = 0.5$ and $\hat{k}_1 = 3.0$, $\hat{k}_2 = 3.5$, $\hat{k}_3 = 4.5$.

..............., ———, ———, ——— and –·–·–·, respectively. From Table 4, one sees that the values of $\bar{\omega}_i (i = 1$ to $5)$ obtained from the present method (NAM) shown in 1st row of results are very close to the corresponding ones obtained from FEM shown in 2nd row of results. From Tables 4 and 1, one also sees that the lowest three natural frequencies of the loaded beam as shown in Fig. 6 are very close the associated natural frequencies of the three attached spring–mass systems with respect to the static beam, respectively; in addition, one has $\bar{\omega}_i \approx \omega_{b,i-3}$ with $i = 4,5$, where ω_{bi} denotes the ith natural frequency of the bare beam (i.e., $p = 0$) as one may see from the 3rd and 4th rows of results of Table 4. All the last phenomena agree with those appearing in the single-span and two-span beams carrying single or multiple spring–mass systems studied in the previous subsections.

4.4. Free vibration analysis of a four-span beam carrying three spring–mass systems

The *four-span* beam carrying three spring–mass systems studied in this subsection is shown in Fig. 8, where the locations of the *three* intermediate pinned supports are at $\bar{x}_1 = 0.3\ell$, $\bar{x}_2 = 0.5\ell$, and $\bar{x}_3 = 0.7\ell$, while the locations and the non-dimensional parameters of the three spring–masses systems are exactly identical to those given in the last Section 4.3. The lowest five natural frequencies of the loaded beam are

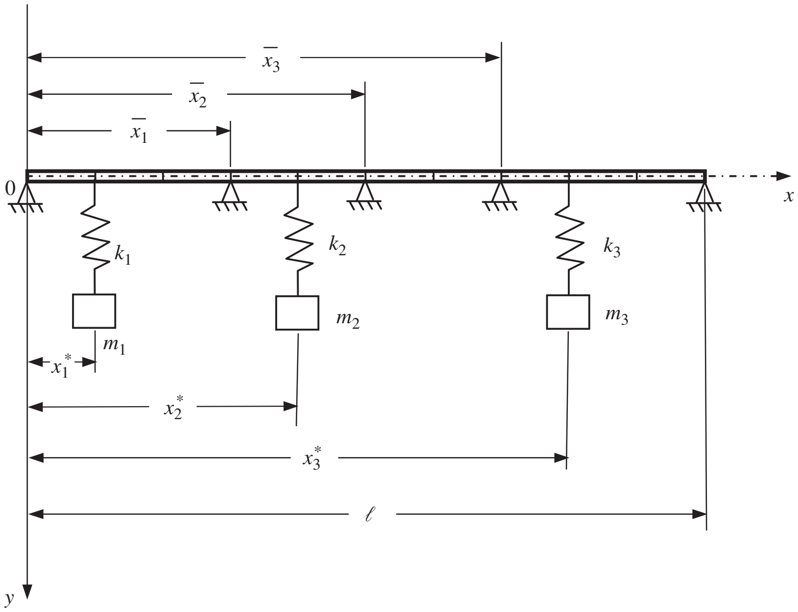

Fig. 8. A four-span pinned–pinned beam (with three intermediate supports at $\bar{x}_1 = 0.3\ell$, $\bar{x}_2 = 0.5\ell$ and $\bar{x}_3 = 0.7\ell$) carrying three intermediate spring–mass systems located at $x_1^* = 0.1\ell$, $x_2^* = 0.4\ell$ and $x_3^* = 0.8\ell$.

Table 5
Influence of the three intermediate spring–mass systems on the lowest five natural frequencies of the four-span pinned–pinned beam shown in Fig. 8

No. of spring–mass system, p	Methods	Natural frequencies (rad/s)				
		$\bar{\omega}_1$ (or ω_{b1})	$\bar{\omega}_2$ (or ω_{b2})	$\bar{\omega}_3$ (or ω_{b3})	$\bar{\omega}_4$ (or ω_{b4})	$\bar{\omega}_5$ (or ω_{b5})
3	Present	192.5744	219.3417	248.6328	8595.0801	9016.0676
	FEM	192.5746	219.3419	248.6329	8595.1227	9016.1159
0	Present	8590.6942	9011.9071	19750.4674	26562.3933	32882.0445
	FEM	8590.7349	9011.9541	19750.5396	26562.4946	32882.1917

Note: $\bar{\omega}_i$ denotes the ith natural frequency of ‘‘loaded’’ beam and ω_{bi} denotes that of the ‘‘bare’’ beam.

shown in the 1st and 3rd rows of results of Table 5 and the associated five mode shapes are shown in Fig. 9. The legend for the curves of Fig. 9 is the same as that of Fig. 7. For comparison, the lowest five natural frequencies of the ‘‘bare’’ beam (cf. Fig. 8) are also listed in the 2nd and 4th rows of results of Table 5. From the last table one sees that the values of $\bar{\omega}_i (i = 1 \text{ to } 5)$ obtained from the present method (NAM) shown in the 1st and 3rd rows of results are very close to the corresponding ones obtained from FEM shown in the 2nd and 4th rows of results. It is similar to the one-span to three-span beams carrying multiple spring–mass systems studied in the previous subsections that the lowest three natural frequencies of the loaded beam shown in the 1st and 2nd rows of results of Table 5 are very close the associated natural frequencies of the three spring–mass systems with respect to the static beam shown in the final row of Table 1, respectively; in addition, one has the relationship $\bar{\omega}_{3+i} \approx \omega_{bi}$ with $i = 1$, 2, where ω_{bi} denotes the ith natural frequency of the bare beam (i.e., $p = 0$) as one may see from the 3rd and 4th rows of results of Table 5. Furthermore, the lowest three mode shapes shown in Fig. 9 take the same form, because all of them are related to the three spring–mass systems (cf. Table 1) vibrating with respect to the static beam, respectively.

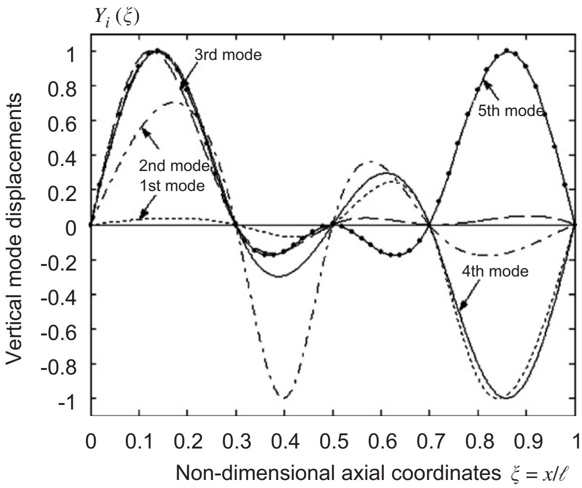

Fig. 9. The lowest five mode shapes of a four-span pinned–pinned beam (with three intermediate supports at $\bar{x}_1 = 0.3\ell$, $\bar{x}_2 = 0.5\ell$ and $\bar{x}_3 = 0.7\ell$) carrying three intermediate spring–mass systems located at $\bar{x}_3 = 0.1\ell$, $x_2^* = 0.4\ell$ and $x_3^* = 0.8\ell$. The non-dimensional magnitudes of the three spring–mass systems are $\hat{m}_1 = 0.2$, $\hat{m}_2 = 0.3$, $\hat{m}_3 = 0.5$ and $\hat{k}_1 = 3.0$, $\hat{k}_2 = 3.5$, $\hat{k}_3 = 4.5$.

It is noted that the total number of intermediate stations for the present example is $N_i = 6$, including three intermediate spring–mass systems (i.e., $S = 3$) and three intermediate supports (i.e., $T-2 = 3$). Thus, according to Eq. (41), the total number of equations for the integration constants is $q = 4(T-2)+5S+4 = 31$. In other words, the order of the overall coefficient matrix [B] is 31×31. For the uniform pinned–pinned beam with one intermediate pinned support and one intermediate spring–mass system as shown in Fig. 4, its overall coefficient matrix [B] is a 13×13 square matrix given in the appendix of this paper. Since it is too lengthy to write the last explicit-form 13×13 overall coefficient matrix [B], the classical explicit analytical methods will suffer much more difficulty for writing the explicit-form 31×31 overall coefficient matrix [B] and calculating the value of associated determinant $|B|$ for the present example.

5. Conclusions

In general, the accuracy of a numerical method is evaluated by comparing its numerical result with the associated "exact" solution. The free vibration analysis of a multi-span beam carrying multiple intermediate spring–mass systems is a practical problem in engineering, but its exact solution for the natural frequencies and mode shapes is not obtained yet. Thus, the information presented in this paper will be significant in this aspect.

For a single-span pinned–pinned beam, if the natural frequency of each spring–mass system with respect to the static beam, ω_p, is smaller than the fundamental frequency of the bare beam, ω_{b1}, then the lowest p natural frequencies of the loaded beam, $\bar{\omega}_p (p = 1, 2, 3, \ldots)$, will be very close to the associated natural frequencies of the attached p spring–mass systems, $\omega_p (p = 1, 2, 3, \ldots)$, respectively. In such a case, one has the relationship $\bar{\omega}_{p+i} \approx \omega_{bi}$ for $i = 1, 2, 3, \ldots$, where p denotes the total number of spring–mass systems attached and ω_{bi} denotes the ith natural frequency of the "bare" beam (i.e., a beam without any spring–mass systems attached). Because the lowest p natural frequencies of the loaded beam, $\bar{\omega}_p (p = 1, 2, 3, \ldots)$, are very close to the associated natural frequencies of the p spring–mass systems with respect to the static beam, ω_p, respectively, their mode shapes (i.e., the corresponding deformations of the loaded beam) take the same form.

From the numerical results of this paper it is found that the last conclusion drawn from the single-span beam is also available for the multi-span beam carrying multiple spring–mass systems.

Appendix

The overall coefficient matrix $[B]$ for a uniform pinned–pinned beam with one intermediate pinned support and one intermediate spring–mass system is found to be

$$[\bar{B}] = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s\theta_1 & c\theta_1 & sh\theta_1 & ch\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\theta_1 & c\theta_1 & sh\theta_1 & ch\theta_1 & 0 & 0 & 0 & 0 & 0 \\ c\theta_1 & -s\theta_1 & ch\theta_1 & sh\theta_1 & -c\theta_1 & s\theta_1 & -ch\theta_1 & -sh\theta_1 & 0 & 0 & 0 & 0 & 0 \\ -s\theta_1 & -c\theta_1 & sh\theta_1 & ch\theta_1 & s\theta_1 & c\theta_1 & -sh\theta_1 & -ch\theta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\theta_1^* & c\theta_1^* & sh\theta_1^* & ch\theta_1^* & -s\theta_1^* & -c\theta_1^* & -sh\theta_1^* & -ch\theta_1^* & 0 \\ 0 & 0 & 0 & 0 & c\theta_1^* & -s\theta_1^* & ch\theta_1^* & sh\theta_1^* & -c\theta_1^* & s\theta_1^* & -ch\theta_1^* & -sh\theta_1^* & 0 \\ 0 & 0 & 0 & 0 & -s\theta_1^* & -c\theta_1^* & sh\theta_1^* & ch\theta_1^* & s\theta_1^* & c\theta_1^* & -sh\theta_1^* & -ch\theta_1^* & 0 \\ 0 & 0 & 0 & 0 & -c\theta_1^* & s\theta_1^* & ch\theta_1^* & sh\theta_1^* & c\theta_1^* & -s\theta_1^* & -ch\theta_1^* & -sh\theta_1^* & \hat{m}_1\Omega \\ 0 & 0 & 0 & 0 & s\theta_1^* & c\theta_1^* & sh\theta_1^* & ch\theta_1^* & 0 & 0 & 0 & 0 & \lambda_1^2 - 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\Omega & c\Omega & sh\Omega & ch\Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s\Omega & -c\Omega & sh\Omega & ch\Omega & 0 \end{bmatrix},$$

where $s\Omega = \sin\Omega$, $c\Omega = \cos\Omega$, $sh\Omega = \sinh\Omega$, $ch\Omega = \cosh\Omega$, $s\theta_1 = \sin\theta_1$, $c\theta_1 = \cos\theta_1$, $sh\theta_1 = \sinh\theta_1$, $ch\theta_1 = \cosh\theta_1$, $\theta_1 = \Omega\bar{\xi}_1$, $\bar{\xi}_1 = \bar{x}_1/\ell$, $s\theta_1^* = \sin\theta_1^*$, $c\theta_1^* = \cos\theta_1^*$, $sh\theta_1^* = \sinh\theta_1^*$, $ch\theta_1^* = \cosh\theta_1^*$, $\theta_1^* = \Omega\,\xi_1^*$, $\xi_1^* = x_1^*/\ell$, $\lambda_1 = \bar{\omega}/\omega_1$, $\omega_1 = \sqrt{k_1/m_1}$, $\bar{\omega} = (\Omega)^2(EI/\bar{m}\ell^4)^{1/2}$.

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